Section 7.4: Integration of Rational Functions by Partial Fractions

Objective: In this lesson, you learn

- ☐ How to integrate any rational function by expressing it as a sum of partial fractions.
- ☐ How to convert a nonrational function to a rational function by an appropriate substitution.

I. Integration of Rational Functions by Partial Fractions.

To integrate any rational function (a ratio of polynomials), express it as a sum of simpler fractions, called **partial fractions**, which we already know how to integrate.

Problem: Evaluate

$x-1$

$$\int \frac{x-1}{x^2-5x+6}\,dx$$

let $u = x^2-5x+6 \Rightarrow du = 2x-5\ dx \Rightarrow \frac{du}{2} = x-\frac{5}{2}\,dx$

$\left(\frac{5}{2}\right)^2 = \frac{25}{4}$ You can see that $\boxed{2x-5 \neq x-1}$

$$x^2-5x+\frac{25}{4}-\frac{25}{4}+6 = \left(x-\frac{5}{2}\right)^2+\left(\frac{-25}{4}+6\right)$$
$$= \left(x-\frac{5}{2}\right)^2+\left(\frac{-1}{4}\right)$$
$$= \left(x-\frac{5}{2}\right)^2-\left(\frac{1}{2}\right)^2$$

let $t = x-\frac{5}{2} \Rightarrow dt = dx$

let $t = \frac{1}{2}\sec\theta$ (You can do it by trig sub.)

OR

$$\frac{x-1}{x^2-5x+6} = \frac{x-1}{(x-2)(x-3)} = \frac{A\,(x-3)}{(x-2)(x-3)} + \frac{B\,(x-2)}{(x-3)(x-2)}$$

$$\frac{x-1}{(x-2)(x-3)} = \frac{A(x-3)+B(x-2)}{(x-2)(x-3)} = \frac{Ax-3A+Bx-2B}{(x-2)(x-3)}$$

$$x-1 = (A+B)x+(-3A-2B) \Rightarrow \begin{array}{l} 3A+3B=3 \\ -3A-2B=-1 \\ \hline \boxed{B=2} \end{array}$$

$A+B=1$
$A+2=1 \Rightarrow \boxed{A=-1}$

$$\frac{x-1}{(x-2)(x-3)} = \frac{A}{(x-2)}+\frac{B}{(x-3)}$$

$$A = \frac{2-1}{2-3} = \frac{1}{-1} = -1$$

$$B = \frac{3-1}{3-2} = \frac{2}{1} = 2$$

$$\int \frac{x-1}{x^2-5x+6}\,dx = \int\frac{-1\,dx}{x-2} + \int\frac{2\,dx}{x-3} = -\ln|x-2|+2\ln|x-3|+C$$

$$x-1 = A(x-3)+B(x-2)$$

$x=2 \Rightarrow \quad 1 = -A+0 \Rightarrow \boxed{A=-1}$

$x=3 \Rightarrow \quad 2 = 0+B \Rightarrow \boxed{B=2}$

General Problem: Evaluate

$$\int f(x)\,dx = \int \frac{P(x)}{Q(x)}\,dx,$$

$a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$

where $P(x)$ (dividend) and $Q(x)$ (divisor) are polynomials.

If the $\deg(P(x)) > \deg(Q(x))$ then (by the long division) there are two polynomials $q(x)$ and $r(x)$ such that

$$f(x) = \frac{P(x)}{Q(x)} = q(x) + \frac{r(x)}{Q(x)}.$$

where $r(x) = 0$ or $\deg(r(x)) < \deg(Q(x))$. The polynomial $q(x)$ is the quotient and $r(x)$ is the remainder produced by the long division process.

If $r(x) = 0$, then $\dfrac{P(x)}{Q(x)}$ is really just a polynomial, so we can ignore that case here.

Now,

$$\int f(x)\,dx = \int \frac{P(x)}{Q(x)} = \int q(x)\,dx + \int \frac{r(x)}{Q(x)}\,dx.$$

We can easily integrate the polynomial $q(x)$, so the general problem reduces to the problem of integrating a rational function $\dfrac{r(x)}{Q(x)}$ with $\deg r(x) < \deg Q(x)$.

So, for the purposes of investigating how to intergrate a rational function we can suppose

$$f(x) = \frac{P(x)}{Q(x)}$$

with $\deg P(x) < \deg Q(x)$.

$x_{1,2} = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Fact about every $Q(x)$:

$Q(x)$ can be factored as a product of linear factors (i.e. of the form $ax + b$) and/or irreducible quadratic forms (i.e. of the form $ax^2 + bx + c$, where $b^2 - 4ac < 0$).

Our strategy to integrate the rational function $f(x)$ is as follows:

- Factor $Q(x)$ into linear and irreducible quadratic factors.
- write $f(x)$ as a sum of partial fractions, where each fraction is of the form

$$\frac{K}{(ax+b)^s} \qquad \text{or} \qquad \frac{Lx+M}{(ax^2+bx+c)^t}$$

- integrate each partial fraction in the sum.

Question: How do we find K, L, and M?

Partial-Fraction Decomposition: **General Techniques**

Case 1: The denominator $Q(x)$ is a product of distinct linear factors, that is, $Q(x) = (a_1x+b_1)(a_2x+b_2)\cdots(a_kx+b_k)$, where no factor is repeated and no factor is a constant multiple of another. In this case, there exists constants $A_1, A_2, \ldots A_k$ such that

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \cdots \frac{A_k}{a_kx+b_k}.$$

Example 1: Evaluate $\int \frac{2x+1}{x^2-1}\,dx$

$Q(x) = x^2-1 = (x-1)(x+1)$

$$\int \frac{2x+1}{x^2-1}\,dx = \int \frac{2x+1}{(x-1)(x+1)}\,dx$$

$$\frac{2x+1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$$

$$A = \frac{3}{2}, \quad B = \frac{-1}{-2} = \frac{1}{2}$$

$$= \int \frac{3/2}{x-1}\,dx + \int \frac{1/2}{x+1}\,dx$$

$$= \frac{3}{2}\ln|x-1| + \frac{1}{2}\ln|x+1| + C.$$

$$\int \frac{1}{x+a}\,dx = \ln|x+a|$$

Example 2: Evaluate $\displaystyle\int \frac{4x^2-3x-4}{x^3+x^2-2x}\,dx$

$$x^3+x^2-2x = x(x^2+x-2) = x(x+2)(x-1)$$

$$\frac{4x^2-3x-4}{x(x+2)(x-1)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-1}$$

$$A = \frac{-4}{-2} = 2, \quad B = \frac{16+6-4}{6} = \frac{18}{6} = 3, \quad C = \frac{-3}{3} = -1$$

$$\int \frac{4x^2-3x-4}{x(x+2)(x-1)}\,dx = 2\int \frac{1}{x}\,dx + 3\int \frac{1}{x+2}\,dx - 1\int \frac{1}{x-1}\,dx$$

$$= 2\ln|x| + 3\ln|x+2| - \ln|x-1| + C.$$

Case 2: The denominator $Q(x)$ is a product of linear factors, some of which are repeated. Suppose the first linear factor (a_1x+b_1) is repeated r times so that $(a_1x+b_1)^r$ occurs in the factorization of $Q(x)$. Then instead of the single term $A_1/(a_1x+b_1)$ (in **case 1**) we would use

$$\frac{A_1}{a_1x+b_1}+\frac{A_2}{(a_1x+b_1)^2}+\cdots\frac{A_r}{(a_1x+b_1)^r}.$$

Example 3: Evaluate $\displaystyle\int \frac{2x-1}{(x-5)^2}\,dx$

$(2x-1) = A(x-5) + B$

$$\left[\frac{2x-1}{(x-5)^2} = \frac{A}{x-5} + \frac{B}{(x-5)^2}\right] * (x-5)^2$$

$$2x-1 = A(x-5)+B$$

$x=5 \Rightarrow \boxed{B=9}$

$x=6 \Rightarrow 12-1 = A+9 \Rightarrow \boxed{A=2}$

$x=0$

$-1=-5A+9$

$-10=-5A$

$\frac{}{-5} \quad \frac{}{-5}$

$A=2$

$$\int \frac{2x-1}{(x-5)^2}\,dx = 2\int\frac{1}{x-5}\,dx + 9\int\frac{1}{(x-5)^2}\,dx$$

$$= 2\ln|x-5| - \frac{9}{x-5} + C.$$

$u = x-5$

$dx = dx$

$\int \frac{1}{u^2}\,du$

$\int u^{-2}\,du$

$= \frac{u^{-1}}{-1}$

$= \frac{-1}{u}$

Example 4: Evaluate $\displaystyle\int \frac{x^3-4x-1}{x(x-1)^3}\,dx$

$$\left(\frac{x^3-4x-1}{x(x-1)^3} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} + \frac{D}{(x-1)^3}\right) * x(x-1)^3$$

$$x^3-4x-1 = A(x-1)^3 + Bx(x-1)^2 + Cx(x-1) + Dx$$

$x=0 \Rightarrow \quad -1 = -A \Rightarrow \boxed{A=1}$

$x=1 \Rightarrow \quad \boxed{-4 = D}$

$x=2 \Rightarrow \quad -1 = 1 + 2B + 2C + 2(-4)$

$\quad 2B+2C=6 \Rightarrow \boxed{B+C=3}$

$x=-1 \Rightarrow \quad 2 = 1\cdot(-2)^3 + B(-1)(-2)^2 + C(-1)(-2) + (-4)(-1)$

$\quad 2 = \overset{-4}{\cancel{-8}} - 2B + 2C \cancel{+4}$

$\quad \frac{6}{2} = \frac{-2B}{2} + \frac{2C}{2} \Rightarrow \boxed{-B+C=3}$

$+$

$2C = 6 \Rightarrow \boxed{C=3}$

$\Rightarrow B=0$

$$\int \frac{x^3-4x-1}{x(x-1)^3}\,dx = \int \frac{1}{x}\,dx + \int \frac{3}{(x-1)^2}\,dx + \int \frac{-4}{(x-1)^3}\,dx$$

$$= \ln|x| + \frac{-3}{x-1} - \frac{4}{-2(x-1)^2} + C$$

$x_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

Case 3: The denominator $Q(x)$ contains irreducible quadratic factors, none of which is repeated. If $Q(x)$ has ax^2+bx+c, where $b^2-4ac<0$, then, in the sum of its partial fractions, the expression $\frac{P(x)}{Q(x)}$ will have terms of the form

$$\frac{Ax+B}{ax^2+bx+c},$$

where A and B are constants to be determined.

$x^2+1=0$

$x^2=-1$

No solution

Example 5: Evaluate $\int \frac{5x^3-3x^2+2x-1}{x^4+x^2}\,dx$

$x^4+x^2 = x^2(x^2+1)$

$$\frac{5x^3-3x^2+2x-1}{x^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Dx+C}{x^2+1}$$

$x^2(x^2+1)$

$x=0 \Rightarrow B = \frac{-1}{1} = -1$

$5x^3-3x^2+2x-1 = Ax(x^2+1) - (x^2+1) + (Dx+C)x^2$

$= Ax^3 + Ax - x^2 - 1 + Dx^3 + Cx^2$

$A=2$

$-3x^2 = -x^2+Cx^2 = (-1+C)x^2$

$-1+C=-3 \Rightarrow C=-2$

$5x^3 = Ax^3+Dx^3 = (2+D)x^3$

$2+D=5 \Rightarrow D=3$

$$= \int \frac{2}{x}\,dx + \int \frac{-1}{x^2}\,dx + \int \frac{3x-2}{x^2+1}\,dx$$

$$= 2\ln|x| + \frac{1}{x} + \frac{3}{2}\int \frac{2x}{x^2+1}\,dx - 2\int \frac{1}{x^2+1}\,dx$$

$+\frac{3}{2}\ln(x^2+1) - 2\tan^{-1}(x) + C$

$(\ln f(x))' = \frac{f'(x)}{f(x)}$

$$x^4 \overline{\big)\, x^4 - 1}$$

Example 6: Evaluate $\displaystyle\int \frac{x^4}{x^4-1}\,dx$

$$\frac{x^4}{x^4-1} = \frac{x^4-1+1}{x^4-1} = \frac{x^4-1}{x^4-1} + \frac{1}{x^4-1}$$
$$= 1 + \frac{1}{x^4-1}$$

$$(x^4-1) = (x^2-1)(x^2+1) = (x-1)(x+1)(x^2+1)$$

$$\left[\frac{1}{x^4-1} = \frac{1}{(x-1)(x+1)(x^2+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Dx+C}{x^2+1}\right](x-1)(x+1)(x^2+1)$$

$A = \frac{1}{4}$, $B = \frac{-1}{4}$,

$$1 = \frac{1}{4}(x+1)(x^2+1) - \frac{1}{4}(x-1)(x^2+1) + (Dx+C)(x^2-1)$$

$$x=0 \Rightarrow 1 = \frac{1}{4} - \frac{1}{4}(-1) + C(-1)$$
$$1 = \frac{1}{2} - C \Rightarrow -C = 1/2 \Rightarrow C = -1/2$$

$$x=3 \Rightarrow 1 = \cancel{\tfrac{1}{4}}(\cancel{4})(10) - \cancel{\tfrac{1}{4}}(\cancel{2})(\cancel{10})^{5} + (3D - \tfrac{1}{2})(8)$$
$$\cancel{1} = \cancel{10}^{1} - \cancel{5} + 24D - \cancel{4}$$
$$24D = 0 \Rightarrow D = 0$$

$$= 1 + \int \frac{1/4}{x-1}\,dx + \int \frac{-1/4}{x+1}\,dx + \int \frac{-1/2}{x^2+1}$$
$$= x + \frac{1}{4}\ln|x-1| - \frac{1}{4}\ln|x+1| - \frac{1}{2}\tan^{-1}(x) + C.$$

Case 4: The denominator $Q(x)$ contains a repeated irreducible quadratic factor. If $Q(x)$ has the $(ax^2+bx+c)^r$, where $b^2-4ac<0$, then, in the sum of its partial fractions, the expression $\frac{P(x)}{Q(x)}$ will have a sum

$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots\frac{A_rx+B_r}{(ax^2+bx+c)^r}.$$

Example 7: Evaluate $\displaystyle\int \frac{dx}{x(x^2+1)^2}$

$$\left[\frac{1}{x(x^2+1)^2}=\frac{A}{x}+\frac{Bx+C}{x^2+1}+\frac{Dx+R}{(x^2+1)^2}\right]\times(x^2+1)^2$$

$$\boxed{A=1}$$

$$1=(x^2+1)^2+(Bx+C)x(x^2+1)+(Dx+R)x$$

$$1=\boxed{x^4}+2\cancel{x}^2+1+\boxed{Bx^4}+B\cancel{x}^2+\cancel{C}x^3+\cancel{C}x+D\cancel{x}^2+Rx$$

$$0=(B+1)x^4 \Rightarrow B+1=0 \Rightarrow B=-1$$

$$0=Cx^3 \Rightarrow C=0$$

$$0=(2+B+D)x^2 \Rightarrow 2-1+D=0 \Rightarrow D=-1$$

$$0=Rx \Rightarrow R=0$$

$$=\int\frac{1}{x}\,dx+\frac{1}{2}\int\frac{-2x}{x^2+1}\,dx+\int\frac{-1}{(x^2+1)^2}\,dx$$

$$=\ln|x|-\frac{1}{2}\ln|x^2+1|-\int\frac{1}{(x^2+1)^2}\,dx \rightarrow x=\tan\theta$$

$$=\ln|x|-\frac{1}{2}\ln|x^2+1|-\frac{1}{2}\left[\tan^{-1}x+\ln|1+x^2|\right]+C$$

II. Rationalizing Substitutions.

Some nonrational functions can be changed into rational functions by means of appropriate substitutions. For example, if an expression of the form $\sqrt[n]{g(x)}$ is in an integrand, then the substitution

$$u = \sqrt[n]{g(x)}$$

may be effective.

Example 8: Evaluate $\int \frac{\sqrt{x+4}}{x}\,dx$

let $u = \sqrt{x+4}$ $\quad du = \frac{1}{2\sqrt{x+4}}\,dx = \frac{1}{2u}\,dx$

$\boxed{u^2 - 4 = x}$ $\quad \boxed{2u\,du = dx}$

$\int \frac{x}{\sqrt{x+4}}$

$u = x+4$

$du = dx$

$\int \frac{u-4}{\sqrt{u}}\,du$

$\int u^{1/2} - 4u^{-\frac{1}{2}}\,du$

$\frac{2}{3}u^{3/2} - 8u^{1/2} + C$

$$\int \frac{\sqrt{x+4}}{x}\,dx = \int \frac{u}{u^2-4}\cdot 2u\,du$$

$$= \int \frac{2u^2}{u^2-4}\,du = 2\int \frac{u^2-4+4}{u^2-4}\,du$$

$$= 2\int 1 + \frac{4}{u^2-4}\,du$$

$$= 2u + 4\int \frac{1}{u^2-4}\,du$$

$$\frac{1}{u^2-4} = \frac{1}{(u-2)(u+2)} = \frac{A}{u-2} + \frac{B}{u+2}$$

$$A = \frac{1}{4},\quad B = \frac{-1}{4}$$

$$= 2u + \cancel{4}\left[\int \frac{1/\cancel{4}}{u-2}\,du + \int \frac{-1/\cancel{4}}{u+2}\,du\right]$$

$$= 2u + \ln|u-2| - \ln|u+2|$$

$$= 2\sqrt{x+4} + \ln\left|\frac{\sqrt{x+4}-2}{\sqrt{x+4}+2}\right| + C$$

Example 9: Evaluate $\int \frac{\sqrt{x}}{x^2+x}\,dx$

let $u = \sqrt{x} \longrightarrow du = \frac{1}{2\sqrt{x}}\,dx$

$\boxed{u^2 = x}$

$dx = 2u\,du$

$$\int \frac{u}{u^4+u^2} \cdot 2u\,du$$

$$= 2\int \frac{\cancel{u^2}}{\cancel{u^2}(u^2+1)}\,du = 2\tan^{-1}(u) + C$$

$$= 2\tan^{-1}(\sqrt{x}) + C$$

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